Chapter 9

Fixed-Income Returns from Hedge Funds with Negative Fee Structures: Valuation and Risk Analysis

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The traditional fixed-income asset class has generated very low returns in recent years. Furthermore, due to long-term market trends it is arguably perceived by investors to be riskier and less diversifying than it has ever been. This has led to the emergence of new products that are designed to appeal to institutional investors in their quest for finding complementary return streams, particularly for liability driven investment (LDI). These bond-like products are often augmented with equity-like positions in investors' portfolios in an attempt to mitigate risk and generate attractive returns. In this paper, we analyze fee structures that have emerged in the hedge fund industry. In particular, we study structures with 'negative fees,' which give hedge fund investments risk-return profiles that more closely resemble traditional fixed-income investments. We analyze the value and risk-return profiles of these investments, and study the incentives that the fee structures create for fund managers. In this paper we discuss how the employment of judicious fee structures in combination with suitable trading strategies can assist in accommodating the appetite of a wide range of investors. We will present a spectrum of fee structures where investors can pinpoint a region of interest which fulfills their desired payoff profile.

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1. Introduction

According to FitchRatings, the total of sovereign debt with negative yields increased to \$11.7 trillion as of June 27, 2016, up \$1.3 trillion from the total at the end of May.[a] Major institutional investors have approximately 30% to 50% of their assets allocated to fixed income, which makes them increasingly vulnerable to the interest rate environment (OECD [12]). The low-rate environment has also impacted the manner in which hedge fund managers are compensated. Investors accept paying the traditional fees to hedge fund managers only if the underlying trading strategy generates superior returns (or alpha). However, the lukewarm performance of hedge funds in recent years has pressured the fees as investors need to maintain an acceptable share of gross returns to meet their investment thresholds. The low-rate environment has significantly trimmed the short rebates that managers used to receive on their short book resulting in lower performance of trading strategies in general. This has further undermined the acceptability of traditional 2&20 fee structures (see Bloomberg [2]) and has encouraged investors to seek innovative fee methodologies.[b]

Investors' demand for yield, combined with the difficult market environment and the challenges faced by many hedge fund managers in raising assets, has led institutional investors and fund managers to embrace new fee structures featuring an element of downside protection. In these fee structures, commonly referred to as 'first-loss' or 'shared-loss' structures, the fund manager insures a portion of the investor's losses.

There are many variations on the basic framework of the first-loss fee structure, all of which share the following principle: the fund manager provides downside protection by taking the first tranche of losses, and in return the manager receives a higher percentage of upside participation (higher than the standard performance fee in traditional fee structures). For example, a manager may absorb losses up to 10% and in return may be entitled to a 50% monthly performance fee as opposed to a 20% annual performance fee. Investors gain exposure to the hedge fund investment, with the manager taking the first tranche of losses. He and Kou [9] analyze the first-loss fee structure, examining the incentives that it creates for hedge fund managers, as well as its impact on the utility of both investors and managers. They conclude that for some parameter values, the first-loss fee structure can increase the utility of both the investor and the manager, and result in a less risky investment portfolio. However, at the levels of the performance

[a] https://www.fitchratings.com/site/pr/1008156.

[b] The traditional 2&20 fee structure consists of a flat fee of 2% of assets under management together with a performance fee of 20% of net profits.

fee commonly charged in the industry, they find a significant reduction in investor utility.[c] Djerroud *et al.*[3] analyzed the first-loss fee structure using an option-pricing perspective, providing a value of the guarantee offered by the manager to the investor, and compared its value to the performance fee offered by the investor to the manager, providing a way of assessing 'fairness,' which can be used as a benchmark for negotiation of the terms of the fee structure between investor and manager.

In this paper we extend the concept of First Loss by considering a guarantee not just against losses but providing a minimum return guarantee from the manager to the investor. In this regard, the investment starts to look to the investor like a bond with a coupon payment that contains two parts: a fixed one, coming from the return guarantee offered by the hedge fund, and a variable one, arising from the performance of the hedge fund investment net of performance fees. Figure 1 illustrates a spectrum of fee structures from the traditional to the first-loss family of structures and beyond. In the traditional fee structure, also known as '2&20', the investor return varies with the performance of the hedge fund strategy; the investor can experience periods of losses as seen in the leftmost bar in the figure. A simple first-loss structure involves a higher share of the strategy performance allocated to the manager in return for downside protection for the investor. The investor will be less likely to experience periods of losses under this structure; however, the investor return could be zero. A first variant of the first-loss fee structure is the one in which the investor requires a minimum return coupled with a smaller share of the strategy performance in exchange for a higher performance fee paid to the manager. The rightmost bar illustrates a first-loss structure in which the investor 'swaps' the performance of the strategy on its capital for a promised fixed 'coupon'. We refer to these two last structures as a 'negative fee structure'. From left to right, the upside to the investor is gradually reduced in exchange for downside protection, provided by the fund manager. In addition, the investor is more certain to receive a higher minimum return or a larger 'coupon'. It should be noted that the performance fees on the horizontal axis are for illustration purposes only, and the size of the fixed coupon is dependent on the underlying strategy.

The remainder of the paper is structured as follows. The second section discusses hedge fund fee structures. The third section analyzes negative fee structures from an option pricing perspective under a regime-switching model using

[c]It should be noted that investor capital brings further benefits to a hedge fund, beyond simply the fees accrued, such as the reputational benefit of having more assets under management.

Fig. 1. Schematic representation of hedge fund fee structures from traditional to first-loss fee structures.

risk-neutral valuation. The fourth section analyzes the risks of the investor's returns under a negative fee structure, now using the real-world measure. The fifth section concludes.

2. Hedge fund fee structures: From traditional fee structures to negative fees

2.1. *Traditional fee structures*

Traditionally, a hedge fund manager charges two types of fees to the fund investors:

- A fixed management fee, usually ranging from 1% to 2% of net asset values.

- A performance fee, most commonly equal to 20% of net profits obtained by the fund.

In this paper we assume a single investor and a single share issued by the fund. The extension to the case of multiple investors and multiple shares is straightforward. Although fees are paid according to a determined schedule (usually monthly or quarterly for management fees and annually for performance fees) we will assume a single payment at the end of a fixed term T.

The initial fund supplied by the investor is denoted by X_0. The hedge fund manager then invests the fund assets to create the future gross values X_t, for $t > 0$. The gross fund value X_t is split between the investor's share I_t (the net asset value) and the manager's fee M_t:

$$X_t = I_t + M_t.$$

At time 0, $I_0 = X_0$ and $M_0 = 0$.

There are countless variations on the basic framework, including hurdles, clawbacks, etc. (for more details on first-loss arrangements see Banzaca[1]). We will ignore these and assume the commonly used version of a management fee equal to $m \cdot X_0$ (m represents a fixed percentage of the initial investment by the investor), and a performance fee of $\alpha \cdot (X_T - (1+m)X_0))_+$, so that the performance fee is payable only when the investor's return is positive, and is zero when it is negative. Hence, the manager's payoff due to fees is:

$$M_T = m \cdot X_0 + \alpha \cdot (X_T - (1+m)X_0)_+$$

In other words, while the management fee is a fixed future liability to the investor, the performance fee is a contingent claim on the part of the manager. As a consequence, we will be pricing the management fee simply as a fixed guaranteed fee with a predetermined future cash value, and we will be valuing the performance fee as the value of a certain call option. In our setting, we will assume a regime-switching process for the invested assets X_t, which allows us to value the performance fee using known results. It is worth mentioning that hedge fund managers can speculate on volatility, credit risks, etc. and in contrast to traditional money managers, they can go long and short. The diversity in investment styles and the different levels of gross and net exposure that they can employ could result in leptokurtic returns, for example through frequent large negative returns in the left tail of the return distribution. Generalization of the current framework to other models of hedge fund returns, for example using stochastic volatility by employing generalized autoregressive conditional heteroskedasticity (GARCH) models, could be a subject for future research.

From a business perspective, it is important to note that the investor has a say in the fees paid to the fund manager: sometimes, as in the case of managed account investments, through a direct negotiation of the fees, at other times, such as in a normal fund structure, through the right not to invest in the fund in the first place. However, when it comes to the choice of the portfolio, the manager has full discretion, within the limits existing in the offering memorandum, without seeking investors' permission or input. This consideration will play a role when we try to extrapolate the results of this article to real investment situations.

2.2. *From first-loss to negative first-loss fee structure*

While the first-loss fee structure protects investors from downside moves in the market, if the manager does not generate returns the investor does not make any profits. A negative fee structure results from modifying the first-loss structure to provide a fixed level of promised return to investors, while maintaining some level of downside protection. The cost of this bond-like return for investors is the increase of the performance fee it pays the manager; we refer to this framework as the 'High-yield bond like' framework. In the limit, the investor has a guaranteed return and pays 100% of the performance beyond the guarantee to the manager. As such, the return profile provided to the investor resembles that of an investor in an asset-backed security, with the underlying portfolio being the assets of the hedge fund; we refer to this framework as the 'swap' framework. In the swap framework, at the end of each period, all returns generated by the strategy are allocated to the investor up to the 'return hurdle' which is negotiated with the hedge fund manager. The remaining returns above the return hurdle are fully allocated to the manager as a performance fee. If the fund return is less than the return hurdle, the manager's deposit is used to make up the difference. In subsequent periods, profits are first used to replenish the manager's deposit, before either the investor's return or the performance fee is paid.

A close look at the negative fee structure reveals that the positions of the investor and the hedge fund manager can be formulated as portfolios of options. The first-loss fee structure was analyzed from an option-pricing perspective using the Black–Scholes model in Djerroud *et al.* [3]. In the next section, we extend that analysis to the negative fee structure under a regime-switching model. Given the bond-like payoff of the negative fee structure, this setting is very similar to the classical Merton model for credit risk (see Merton [11]), with the difference coming from the additional downside protection provided to investors by the hedge fund manager.

Denoting the return threshold by H, the payoff functions of the investor and the manager at the terminal time T are respectively:

$$I_T = \begin{cases} X_0(1+H) & \text{when } (X_T - HX_0) \geq (1-c)X_0 \\ X_T + cX_0 & \text{when } (X_T - HX_0) < (1-c)X_0 \end{cases}$$

$$M_T = \begin{cases} X_T - X_0(1+H) & \text{when } (X_T - HX_0) \geq (1-c)X_0 \\ -cX_0 & \text{when } (X_T - HX_0) < (1-c)X_0 \end{cases}$$

Writing these payoff functions more compactly, we obtain:

$$\begin{aligned} I_T &= X_0(1+H) - ((1-c)X_0 - X_T + HX_0)+ \\ M_T &= X_T - X_0(1+H) + ((1-c)X_0 - X_T + HX_0)+ . \end{aligned} \tag{1}$$

From the above formulas, we see that the investor (manager) has a short (long) position in a put option on the fund assets, with strike price $(1-c)X_0 + HX_0$. Risk-neutral valuation can be applied to derive the price of the positions.[d]

In particular, the value of the investor's position is:

$$V_I(0) = \exp(-rT)X_0(1+H) - P(X_0, T, (1-c)X_0 + HX_0, r)$$

where $P(X, T, K, r)$ is the price of a put option on a non-dividend paying asset with current value of the underlying X, time to expiration T, strike price K, and where the risk-free interest rate is r. The above framework can be easily extended to the case in which the investor receives a portion of the excess return above the return threshold H.

3. Pricing the payoffs

We assume.a regime-switching model, in which the coefficients of a diffusion process for the value of the hedge funds assets themselves follow continuous-time Markov chains. Regime-switching models have found many applications in finance since the seminal work of Hamilton.[7] They are able to reproduce many features of real-world return distributions, including skewness, volatility clustering, and fat tails. For applications of regime-switching models to insurance products with investment guarantees, similar in sprit to the hedge-fund guarantees considered in this paper, see Hardy,[8] For many other financial applications, see the papers in the volumes Mamon and Elliottt,[10] and Zeng and Wu.[13]

[d]It should be noted that, similarly to Merton,[11] some of the assumptions used to justify arbitrage-free pricing methods do not hold in practice in the context in which we are applying the model here. In particular, it is typically not possible for the investor to trade in (or even directly observe) the hedge fund assets X_t.

We assume the regime is governed by a finite state continuous-time Markov chain $\varepsilon(t)$ with state space $\mathbb{S} = \{1,2\}$, where state 1 represents the 'normal' regime and state 2 represents the 'stress' regime. The generator of $\varepsilon(t)$ is the matrix:

$$Q = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix},$$

where λ_1 and λ_2 are the transition rates of leaving states 1 and 2 respectively. The value of the hedge fund assets X_t follows a geometric Brownian motion, except that the coefficients of X_t change with the regime:

$$dX_t = \mu_{\varepsilon(t)} X_t dt + \sigma_{\varepsilon(t)} X_t dZ_t$$

where Z_t is a standard Brownian motion, independent of $\varepsilon(t)$, and $\mu_{\varepsilon(t)}$ and $\sigma_{\varepsilon(t)}$ are constants in each state. For simplicity, when $\varepsilon(t) = 1,2$, we use μ_1, μ_2 and σ_1, σ_2 to denote the growth rates and volatilities in each regime. Finally, the risk-free asset B satisfies $B_t = e^{rt}$. The value of the investor's position can be determined using an expectation under a risk-neutral measure (see Elliott *et al.* [4]) to be:

$$V_I^i = \mathbb{E}_{\mathbb{Q}}[I(T)|\varepsilon(0) = i]. \tag{2}$$

Then, we have

$$V_I^i = \exp -rT(1+H) - P_i(X_0, T, (1-c)X_0 + HX_0, r) \tag{3}$$

where $P_i(X,T,K,r) = \mathbb{E}_{\mathbb{Q}}[e^{-rT}(K - S_T)_+|\varepsilon(0) = i], i = 1,2$ is the European put option price under the Markov-modulated geometric Brownian motion model. Moreover, from Guo [6] and Fuh *et al.*, [5] we obtain:

$$\begin{aligned} P_i(X,T,K,r) &= \mathbb{E}_{\mathbb{Q}}[e^{-rT}(K - S_T)_+|\varepsilon(0) = i] \\ &= e^{-rT} \int_0^{K-1} \int_0^T \frac{y}{K-y} \phi(\ln(K-y), m(t), v(t) f_i(t,T) dt dy \end{aligned} \tag{4}$$

where:

$$\begin{aligned} m(t) &= \ln(X) + (rT - \frac{1}{2}v(t)), \\ v(t) &= (\sigma_1^2 - \sigma_2^2)t + \sigma_1^2 T, \\ f_1(t,T) &= e^{-\lambda_1 T}\delta_0(T-t) + e^{-\lambda_2(T-t)-\lambda_1 t}[\lambda_1 I_0(2(\lambda_1\lambda_2 t(T-t))^{1/2}) \\ &\quad + (\frac{\lambda_1\lambda_2 t}{T-t})^{1/2} I_1(2(\lambda_1\lambda_2 t(T-t))^{1/2})], \\ f_2(t,T) &= e^{-\lambda_2 T}\delta_0(t) + e^{-\lambda_2(T-t)-\lambda_1 t}[\lambda_2 I_0(2(\lambda_1\lambda_2 t(T-t))^{1/2}) \\ &\quad + (\frac{\lambda_1\lambda_2(T-t)}{t})^{1/2} I_1(2(\lambda_1\lambda_2 t(T-t))^{1/2})]. \end{aligned}$$

where $\phi(x, m(t), v(t)$ is the normal density function with mean $m(t)$ and variance $v(t)$, I_0 and I_1 are the modified Bessel functions,

$$I_a(z) = (\frac{z}{2})^a \sum_{k=0}^{\infty} \frac{(a/2)^{2k}}{k!\Gamma(k+a+1)}. \tag{5}$$

and δ_0 is a delta function with a mass at 0.

Figures 2, 3 and 4 illustrate the sensitivity of the value of the investor's payoff to the model parameters. Figure 2 is generated assuming that the market is initially in the normal state ($\varepsilon(0) = 1$). Figure 3 repeats the analysis assuming that the market is initially in the stressed state ($\varepsilon(0) = 2$). Finally, Figure 4 assumes that $\varepsilon(0)$ is random, generated according to the stationary distribution of the Markov chain $\varepsilon(t)$, i.e., $\varepsilon(0) = 1$ with probability $\pi_1 = \lambda_2/(\lambda_1 + \lambda_2)$, and $\varepsilon(0) = 2$ with

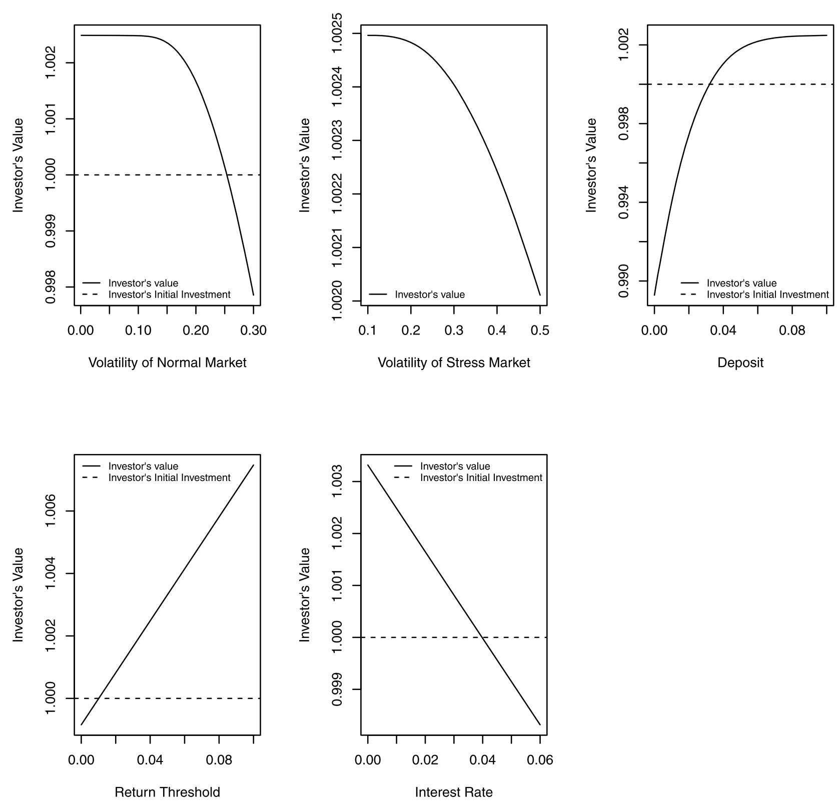

Fig. 2. Sensitivity of the value of the investor's payoff to various parameters, given that the market is initially in the normal state ($\varepsilon(0) = 1$). Benchmark parameter values are $T = \frac{1}{12}$ (the investment horizon is one month), $c = 10\%$ (the manager deposit), $r = 1\%$ (annual risk-free interest rate), $\sigma_1 = 10\%$ (the annual volatility in normal market), $\sigma_2 = 20\%$ (the annual volatility in a stressed market), $\lambda_1 = 1$ (the transition rate in a normal market), $\lambda_2 = 12$ (the transition rate in a stressed market), $X_0 = 1$ (the initial investment), and $H = 4\%$ (the annual return threshold).

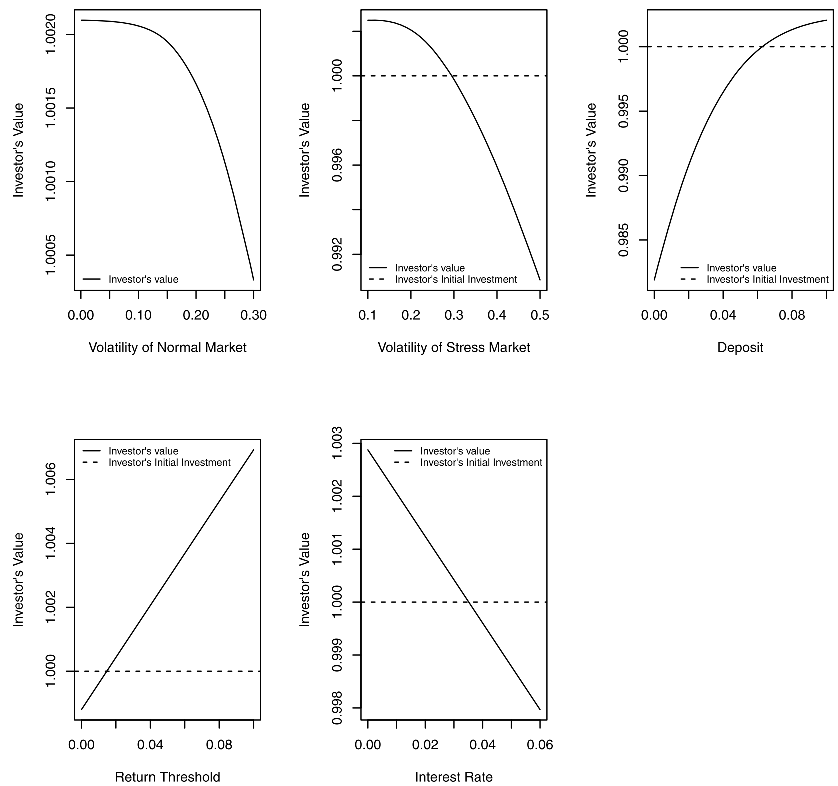

Fig. 3. Sensitivity of the value of the investor's payoff to various parameters, given that the market is initially in the stressed state($\varepsilon(0) = 2$). Benchmark parameter values are $T = \frac{1}{12}$ (the investment horizon is one month), $c = 10\%$ (the manager deposit), $r = 1\%$ (annual risk-free interest rate), $\sigma_1 = 10\%$ (the annual volatility in a normal market), $\sigma_2 = 20\%$ (the annual volatility in a stressed market), $\lambda_1 = 1$ (the transition rate in a normal market), $\lambda_2 = 12$ (the transition rate in a stressed market), $X_0 = 1$ (the initial investment), and $H = 4\%$ (the annual return threshold).

probability $\pi_2 = 1 - \pi_1$. The parameters are set to $T = \frac{1}{12}$ (the investment horizon is one month), $c = 10\%$ (the manager deposit), $r = 1\%$ (annual risk-free interest rate), $X_0 = \$1$ (the initial investment), and $H = 4\%$ (annual return threshold). The volatility and transition rate in a normal market are $\sigma_1 = 10\%$ and $\lambda_1 = 1$, while the corresponding parameters in a stressed market are $\sigma_2 = 20\%$ and $\lambda_2 = 12$.

The same basic patterns emerge when looking at the three sets of figures. The volatility sub-figures show that the value of the investor's position is generally a decreasing function of the volatility parameters of the underlying fund. As the volatility becomes very large, the value of the investor's position starts to decline steeply as the hedge fund's put option (in which the investor has a short position) becomes more valuable. The deposit sub-figures (varying c) illustrate that, as expected, the value of the investor's position is an increasing function of the deposit

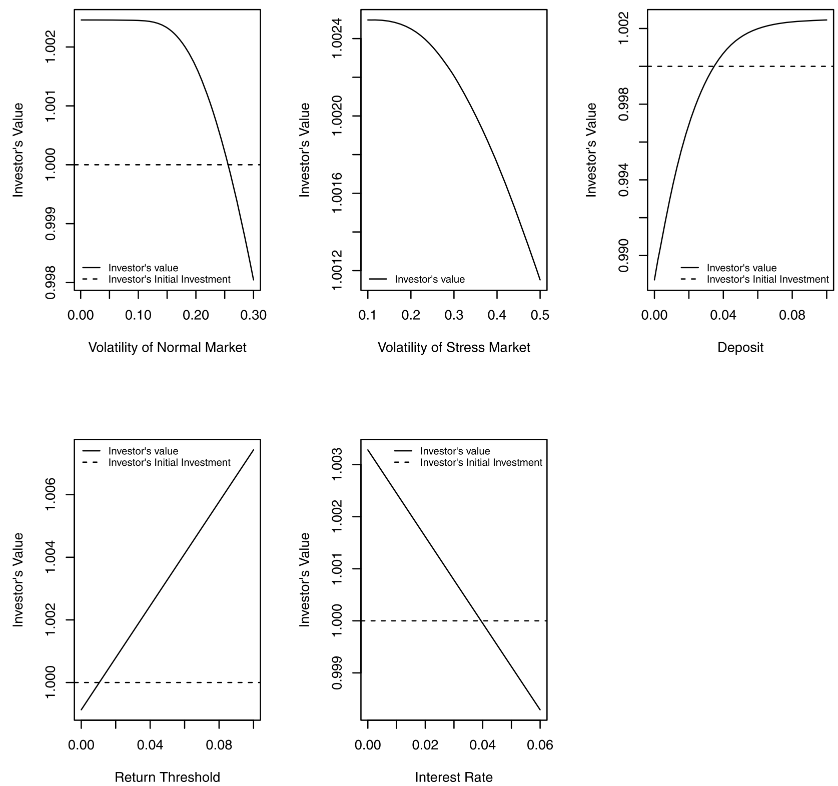

Fig. 4. Sensitivity of the value of the investor's payoff to various parameters, given that $\varepsilon(0)$ is chosen randomly from its stationary distribution. Benchmark parameter values are $T = \frac{1}{12}$ (the investment horizon is one month), $c = 10\%$ (the manager deposit), $r = 1\%$ (annual risk-free interest rate), $\sigma_1 = 10\%$ (the annual volatility in a normal market), $\sigma_2 = 20\%$ (the annual volatility in a stressed market), $\lambda_1 = 1$ (the transition rate in a normal market), $\lambda_2 = 12$ (the transition rate in a stressed market), $X_0 = 1$ (the initial investment), and $H = 4\%$ (the annual return threshold).

level. The return threshold sub-figures show the intuitive monotonic relationship between the value of the investor's position and the return threshold. The value of the investor's position is also a decreasing function of the risk-free rate r, in accord with the bond-like nature of the investor's payoff.

The value of the investor's payoff is lower in a stressed market than in a normal market. The stressed market starts with a higher volatility ($\sigma_2 > \sigma_1$), thus increasing the value of the put option in which the investor has a short position. It is further important to note that given the short time horizon ($T = 1/12$), there is a significant probability that the market will remain in the high volatility, stressed regime over the entire life of the contract. For longer-lived contracts, the discrepancy between the investor's values given that the market is in either the stressed or normal state will be less pronounced. Finally, we note that the

figures for when the initial market state is random with distribution equal to the stationary distribution of $\varepsilon(t)$ are close to those for which the market is started in the normal state. This is due to the fact that with our benchmark parameters, the stationary distribution places a high probability of the market being in a normal state ($\pi_1 = 12/13 \approx 92.3\%$), as is common in financial applications of regime-switching models (see the references cited above).

For each figure, one can look for the point where the curve crosses the value 1.0 (if it exists). This allows us to identify the parameter values for which the contract (in terms of risk-neutral valuation in the regime-switching model) favors either the investor or the manager. Parameter values for which the curve is above 1.0 show that the contract favors the investor, while for parameter values where the curve is below 1.0 the contract favors the manager. The point at which the curve crosses 1.0 is the break-even, or indifference, point.

4. Risk analysis of the investor's position as a bond

As noted above, the position of the investor is analogous to a bond, with a promised return of H (received if the hedge fund assets perform sufficiently well). In the event of default, there is a random amount of recovery (again determined by the level of the hedge fund assets). In this section, we examine the properties of the investor's payoff from the perspective of this analogy with a fixed income investment. In particular, we compute default probabilities and expected recovery rates.

In this section, we consider the manager's and investor's expected payoff under real world measure. In order to obtain numerical results, one can discretize the Markov-modulated geometric Brownian motion process as follows:

$$X_{t+\Delta t} = X_t + \mu_{\tilde{\varepsilon}(t)} X_t \Delta t + \sigma_{\tilde{\varepsilon}_t} X_t \sqrt{\Delta t} \cdot \eta_t$$
$$R_t := \frac{X_{t+\Delta t}}{X_t} - 1 = \mu_{\tilde{\varepsilon}(t)} \Delta t + \sigma_{\tilde{\varepsilon}(t)} \sqrt{\Delta t} \cdot \eta_t$$

where the η_t are i.i.d. standard normal random variables, and $\tilde{\varepsilon}(t)$ is a discretized version of the continuous time Markov chain $\varepsilon(t)$, with transition matrix:

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}.$$

where p is the probability of transitioning from state 1 to state 2, and q is the probability of transitioning from state 2 to state 1. The stationary distribution for this Markov chain is $\pi_0 = q(p+q)^{-1}$, $\pi_1 = p(p+q)^{-1}$.

We simulated 5000 scenarios of the returns of the hedge fund using the above model with an annual time horizon and daily time steps. Recall that the payoffs to the hedge fund manager ($\tilde{M}$) and investor ($\tilde{I}$) for the traditional fee structure are:

$$\tilde{M}(T) = \begin{cases} \alpha(X_T - X_0), & X_T \geq X_0, \\ 0, & X_T < X_0. \end{cases}$$

$$\tilde{I}(T) = \begin{cases} X_0 + (1-\alpha)(X_T - X_0), & X_T \geq X_0, \\ X_T, & X_T < X_0. \end{cases}$$

and the payoffs for the negative loss structure are:

$$M(T) = \begin{cases} X_T - X_0(1+H), & X_T \geq (1-c+H)X_0, \\ -cX_0, & X_T < (1-c+H)X_0. \end{cases}$$

$$I(T) = \begin{cases} X_0(1+H), & X_T \geq (1-c+H)X_0, \\ X_T + cX_0, & X_T < (1-c+H)X_0. \end{cases}$$

Table 1 presents simulated expected returns and standard deviations (with standard errors of the estimates in parentheses) for both the traditional fee structure and the negative fee structure. The volatility in normal markets is set to $\sigma_1 = 0.1$, while in stressed markets it is $\sigma_2 = 0.2$. We consider two different possible growth rates for each state, $\mu_1 = 0.1, 0.15$ and $\mu_2 = -0.05, 0.0$. The probability p is set to 0.01, while q is set to 0.05, indicating a high level of persistence in both states, and a stationary distribution of $(\frac{5}{6}, \frac{1}{6})^T$. In this section, we assume that the initial state of the Markov chain is 1 (normal market). We see that for investors,

Table 1. Expected payoffs (and standard deviations in parentheses) for the traditional fee structure and negative fee structure. $X_0 = 1, T = 1, \alpha = 20\%, p = 0.01, q = 0.05, \sigma_1 = 0.1, \sigma_2 = 0.2$, $N_{sim} = 5000, H = 0.04$, and $c = 0.1$.

(μ_0, μ_1)	$\mathbb{E}_{\mathbb{P}}[\tilde{M}(T)]$	$\mathbb{E}_{\mathbb{P}}[\tilde{I}(T)]$	$\mathbb{E}_{\mathbb{P}}[M(T)]$	$\mathbb{E}_{\mathbb{P}}[I(T)]$
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.0209 (0.0003)	1.0641 (0.0016)	0.0526(0.0017)	1.0323(0.0003)
$\mu_0 = 0.15$, $\mu_1 = -0.05$	0.0296 (0.0004)	1.1077 (0.0017)	0.097 (0.0019)	1.036 (0.0003)
$\mu_0 = 0.1$, $\mu_1 = 0$	0.0218 (0.0003)	1.069 (0.0016)	0.063 (0.0017)	1.034 (0.0003)
$\mu_0 = 0.15$, $\mu_1 = 0$	0.0308 (0.0003)	1.1142 (0.0016)	0.1081 (0.0019)	1.0369(0.0002)
	S.D. of $\tilde{M}(T)$	S.D. of $\tilde{I}(T)$	S.D. of $M(T)$	S.D. of $I(T)$
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.0215 (0.0003)	0.1121 (0.0013)	0.1202 (0.0013)	0.0269 (0.0009)
$\mu_0 = 0.15$, $\mu_1 = -0.05$	0.0251 (0.0003)	0.1167 (0.0012)	0.1345 (0.0014)	0.0178 (0.0008)
$\mu_0 = 0.1$, $\mu_1 = 0$	0.0218 (0.0002)	0.1118 (0.0011)	0.1216 (0.0012)	0.0244(0.0009)
$\mu_0 = 0.15$, $\mu_1 = 0$	0.0247 (0.0003)	0.1131 (0.0012)	0.1312 (0.0013)	0.0169(0.0009)

expected payoffs are slightly higher for the traditional fee structure, but standard deviations are also significantly higher in this case. This is consistent with the analogy that the negative fee structure more closely resembles a fixed income investment, while the traditional fee structure gives a more 'equity-like' payoff. In contrast, the manager's expected payoff and standard deviation are higher under the negative fee structure, and lower under the traditional fee structure. As is to be anticipated, expected payoffs are larger when the growth parameters are larger; the standard deviations of payoffs do not change significantly when the μ_i's are varied.

Tables 2 and 3 repeat the analysis with $p=0.1, q=0.05$, and $p=0.01, q=0.1$ respectively. Comparing to the figures in Table 1, Table 2 was generated assuming a significantly higher (by a factor of 10) probability of transitioning from the normal state to the stressed state, and Table 3 was generated assuming a significantly higher (by a factor of 2) probability of transitioning from the stressed state to the normal state. The stationary distribution for the simulation in Table 2 is $(\frac{1}{3}, \frac{2}{3})^T$ (so that the 'stressed' state is more prevalent), while the stationary distribution for the simulation in Table 3 is $(\frac{10}{11}, \frac{1}{11})^T$. The expected returns and standard deviations of the different payoff structures appear to be relatively insensitive to the choices of the parameters p, q.

Given the similarity of the investor's payoff to the payoff of a fixed income investment, it is interesting to examine the probability of default (i.e. the probability that the investor's return will be lower than the promised hurdle rate H), and the

Table 2. Expected payoffs (and standard deviations in parentheses) for the traditional fee structure and negative fee structure. $X_0 = 1, T = 1, \alpha = 20\%, p = 0.1, q = 0.05, \sigma_1 = 0.1, \sigma_2 = 0.2$, $N_{sim} = 5000, H = 0.04$, and $c = 0.1$.

(μ_0, μ_1)	$\mathbb{E}_{\mathbb{P}}[\tilde{M}(T)]$	$\mathbb{E}_{\mathbb{P}}[\tilde{I}(T)]$	$\mathbb{E}_{\mathbb{P}}[M(T)]$	$\mathbb{E}_{\mathbb{P}}[I(T)]$
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.0163(0.0004)	1.0057(0.0022)	0.0164(0.0020)	1.0056(0.0009)
$\mu_0 = 0.15$, $\mu_1 = -0.05$	0.0181(0.0004)	1.0201(0.0022)	0.0277(0.0021)	1.0104 (0.0009)
$\mu_0 = 0.1$, $\mu_1 = 0$	0.0197(0.0004)	1.0294(0.0023)	0.0374(0.0021)	1.0117(0.0008)
$\mu_0 = 0.15$, $\mu_1 = 0$	0.0223(0.0004)	1.0452(0.0023)	0.0520(0.0023)	1.0155(0.0008)
	S.D. of $\tilde{M}(T)$	S.D. of $\tilde{I}(T)$	S.D. of $M(T)$	S.D. of $I(T)$
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.0248 (0.0004)	0.1576 (0.0017)	0.1426 (0.0021)	0.0633 (0.0011)
$\mu_0 = 0.15$, $\mu_1 = -0.05$	0.0258 (0.0004)	0.1575 (0.0017)	0.1467 (0.0020)	0.0602 (0.0012)
$\mu_0 = 0.1$, $\mu_1 = 0$	0.0266 (0.0004)	0.1597 (0.0016)	0.1507 (0.0020)	0.0581(0.0011)
$\mu_0 = 0.15$, $\mu_1 = 0$	0.0289 (0.0004)	0.1643 (0.0017)	0.1615 (0.0021)	0.0549 (0.0012)

Table 3. Expected payoffs (and standard deviations in parentheses) for the traditional fee structure and negative fee structure. $X_0 = 1, T = 1, \alpha = 20\%, p = 0.01, q = 0.1, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$, and $c = 0.1$.

(μ_0, μ_1)	$\mathbb{E}_{\mathbb{P}}[\tilde{M}(T)]$	$\mathbb{E}_{\mathbb{P}}[\tilde{I}(T)]$	$\mathbb{E}_{\mathbb{P}}[M(T)]$	$\mathbb{E}_{\mathbb{P}}[I(T)]$
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.0227(0.0003)	1.0771(0.0015)	0.0645 (0.0017)	1.0354(0.0003)
$\mu_0 = 0.15$, $\mu_1 = -0.05$	0.0308(0.0003)	1.1169(0.0015)	0.1095 (0.0018)	1.0382(0.0002)
$\mu_0 = 0.1$, $\mu_1 = 0$	0.0225(0.0003)	1.0768(0.0015)	0.0637 (0.0017)	1.0356(0.0003)
$\mu_0 = 0.15$, $\mu_1 = 0$	0.0324(0.0003)	1.1235(0.0015)	0.1179 (0.0018)	1.0381(0.0002)
	S.D. of $\tilde{M}(T)$	S.D. of $\tilde{I}(T)$	S.D. of $M(T)$	S.D. of $I(T)$
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.0214 (0.0002)	0.1049 (0.0011)	0.1178 (0.0012)	0.0187 (0.0008)
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.0235 (0.0002)	0.1048 (0.0011)	0.1245 (0.0013)	0.0107 (0.0007)
$\mu_0 = 0.1$, $\mu_1 = 0$	0.0212 (0.0002)	0.1042 (0.0011)	0.1172 (0.0012)	0.0192 (0.0009)
$\mu_0 = 0.15$, $\mu_1 = 0$	0.0244 (0.0003)	0.1084 (0.0012)	0.1288 (0.0014)	0.0117 (0.0008)

Table 4. Probabilities of default under different parameter assumptions for the regime-switching model. $X_0 = 1, T = 1, \alpha = 20\%, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$, and $c = 0.1$.

	Probability of Default		
(μ_0, μ_1)	$p = 0.01, q = 0.05$	$p = 0.1, q = 0.05$	$p = 0.01, q = 0.1$
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.1338 (0.0048)	0.3544 (0.0067)	0.0962 (0.0042)
$\mu_0 = 0.15$, $\mu_1 = -0.05$	0.0736 (0.0037)	0.3116 (0.0065)	0.0414 (0.0028)
$\mu_0 = 0.1$, $\mu_1 = 0$	0.1248 (0.0047)	0.2968 (0.0065)	0.0914 (0.0041)
$\mu_0 = 0.15$, $\mu_1 = 0$	0.0586 (0.0033)	0.2604 (0.0062)	0.0430 (0.0029)

recovery rate (i.e. the fraction of the promised amount $X_0(1+H)$ that is expected to be recovered conditional upon default having occurred). Simulation results under the regime-switching model for these quantities are provided in Tables 4 and 5 (with standard errors of the estimates in parentheses). Probabilities of default are quite high, ranging from 18% under the best parameter combination to nearly 30% under the worst parameter set. However, these high probabilities of default are mitigated by very high expected recovery rates, in the range of 95-96%.

4.1. *Impact of the manager's deposit* c

A key parameter for first-loss and negative loss fee structures is the manager's deposit c, as it determines the amount of downside protection provided to the

Table 5. Expected recovery rates under different parameter assumptions for the regime-switching model. $X_0 = 1, T = 1, \alpha = 20\%, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$, and $c = 0.1$.

	Recovery Rate		
(μ_0, μ_1)	$p = 0.01, q = 0.05$	$p = 0.1, q = 0.05$	$p = 0.01, q = 0.1$
$\mu_0 = 0.1$, $\mu_1 = -0.05$	0.9825 (0.0507)	0.9429 (0.0722)	0.9922 (0.0399)
$\mu_0 = 0.15$, $\mu_1 = -0.05$	0.9888 (0.0432)	0.9452 (0.0737)	0.9975 (0.0328)
$\mu_0 = 0.1$, $\mu_1 = 0$	0.9844 (0.0455)	0.9447 (0.0707)	0.9919 (0.0439)
$\mu_0 = 0.15$, $\mu_1 = 0$	0.9866 (0.0469)	0.9460 (0.0713)	0.9964 (0.0374)

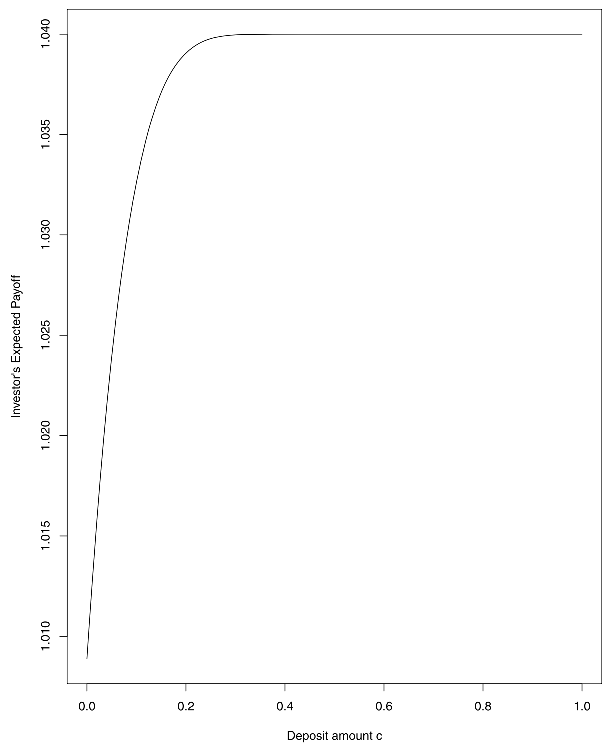

Fig. 5. Investor's expected payoff as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p = 0.01, q = 0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$.

investor by the fund manager (see Djerroud *et al.* [3]). In this section, we investigate the impact of this parameter on the payoffs for the fund investor and manager. Figures 5 and 6 present the expected payoffs of the investor and manager respectively, as the parameter c varies, under the benchmark parameter set used to generate Table 1. We see that for large levels of downside protection, the investor's return quickly approaches the promised value H. For lower levels of insurance,

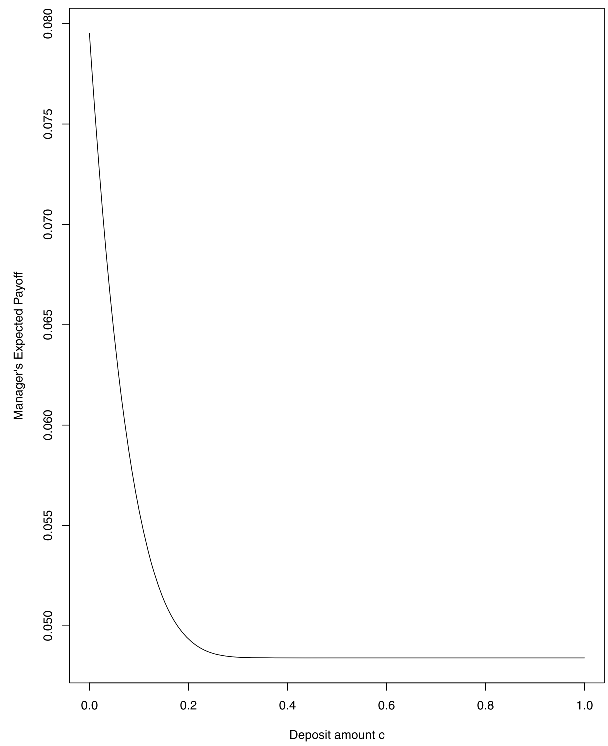

Fig. 6. Manager's expected payoff as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p = 0.01, q = 0.05, \mu_1 = 0.1, \mu_2 = -0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$.

the investor's expected return becomes negative. The manager's expected payoff follows the opposite pattern. Expected payoffs are high for low levels of c, but decrease rapidly as c increases. Similarly, as illustrated in Figures 7 and 8, the volatility of the investor's payoff decreases quickly as the level of downside protection c increases, and the volatility of the manager's payoff increases accordingly. The investor's Sharpe Ratio as a function of c is given in Figure 9 (the risk-free interest rate is set at $r = 1\%$). For very high levels of protection c, the Sharpe ratio grows very quickly (as $H > r$ and a very large level of downside protection virtually guarantees that the investor will receive the return H).

As with many collateralized products, the default and credit risk are intimately related to the market risk and loss quantile of the reference portfolio. In the preceding analysis, we have measured risk using the standard deviations of payoffs and returns. While this is appropriate for normal distributions, the payoffs of the hedge fund manager and investor are non-normal, especially in the context of the regime-switching framework. As a consequence, it is important to consider the tail

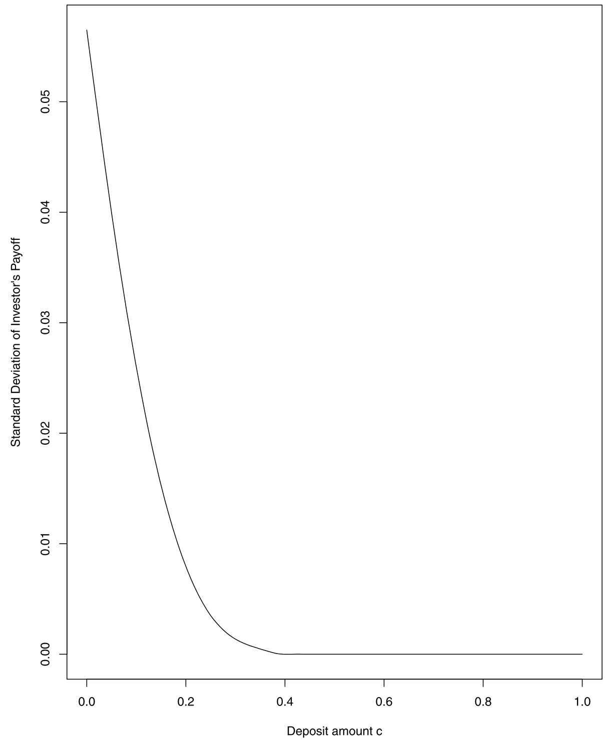

Fig. 7. Standard deviation of the investor's payoff as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p = 0.01, q = 0.05, \mu_1 = 0.1, \mu_2 = -0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$.

risks faced by the investor. We will do this by considering the investor's expected shortfall (also known as conditional Value-at-Risk, or conditional tail expectation), the expectation of losses given that the losses are below a given confidence level of their distribution.

Let

$$L^I = -(I(T) - X_0),$$

so that we have 'positive' loss. Define

$$ES_\beta(L^I) = \mathbb{E}[L^I | L^I \geq VaR_\beta(L^I)]$$

Note that we have a probability mass at the point $-HX_0$. The estimator for expected shortfall is:

$$\widehat{ES}_\beta(L^I) = w \frac{\sum_{i=1}^N \mathbb{I}_{\{L_i^I > \widehat{VaR}_\beta(L^I)\}} L_i^I}{\sum_{i=1}^N \mathbb{I}_{\{L_i^I > \widehat{VaR}_\beta(L^I)\}}} + (1-w)\widehat{VaR}_\beta(L^I)$$

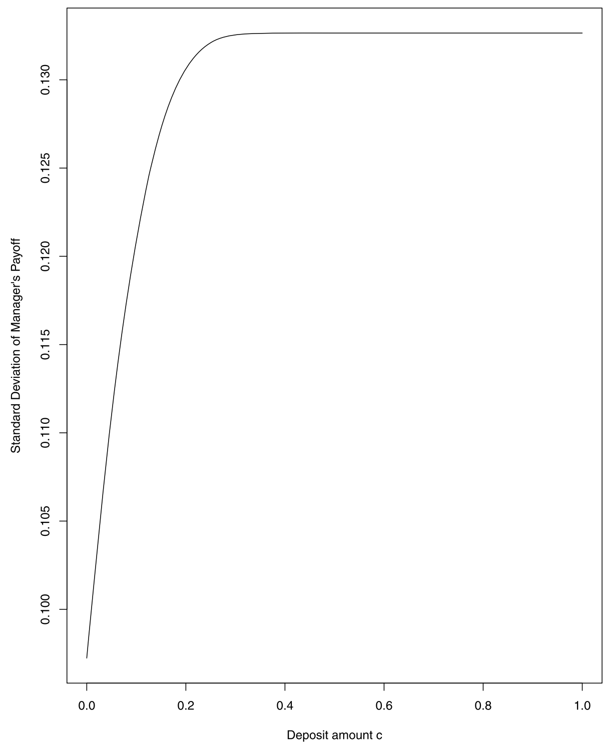

Fig. 8. Standard deviation of the manager's payoff as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p = 0.01, q = 0.05, \mu_1 = 0.1, \mu_2 = -0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$.

where

$$w = \frac{\sum_{i=1}^{N} \mathbb{I}_{\{L_i^I > \widehat{VaR}_\beta(L^I)\}}}{N \cdot (1-\beta)}$$

We increase the number of scenarios in the simulation to 1,000,000 in order to have more scenarios in the tail and a more accurate estimate of expected shortfall. Figure 10 shows the investor's expected shortfall as a function of the manager's deposit c for $\beta = 0.95, 0.99$. As expected, lower levels of the manager's deposit are associated with higher levels of risk. In particular, for manager deposits near our benchmark level of $c = 10\%$, expected shortfall can exceed 20% of the initial investment, indicating significant losses for investors under extreme scenarios. Because of the large number of scenarios used, the confidence intervals for the estimates are quite small (the lengths of the confidence intervals are around 1.5% of the estimated values).

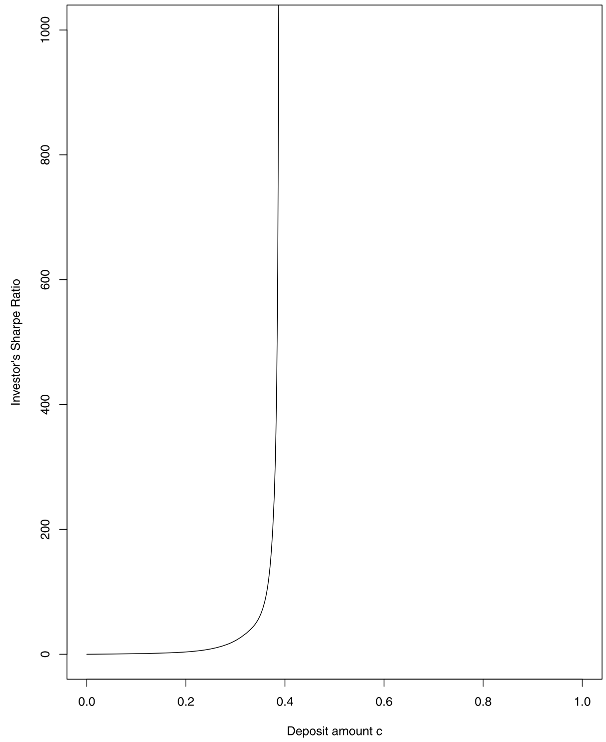

Fig. 9. Sharpe Ratio of the investor's payoff as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p = 0.01, q = 0.05, \mu_1 = 0.1, \mu_2 = -0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 5000, H = 0.04$.

5. Conclusion

Recently, market pressures have led to the introduction of innovative hedge fund fee structures, in which the fund manager receives higher performance fees in return for providing downside protection to fund investors. These arrangements are referred to by the general name of first-loss fee structures. An extreme version is the negative fee structure, in which the manager receives all profits above a pre-defined hurdle rate, and for which the investor's position resembles that of an investment in an asset-backed security, with the underlying assets being the hedge fund's portfolio. In this paper we analyzed the negative fee structure in a regime-switching model, both by pricing it using risk-neutral valuation, and performing a risk analysis under the real-world measure (including examining the probability of default and expected recovery rate).

There are a number of important questions that could be considered for future research. The fee structure could be analyzed under other mathematical models, including those that allow more general stochastic behavior of volatility. The incentives of both the manager, in terms of the structuring of the hedge fund port-

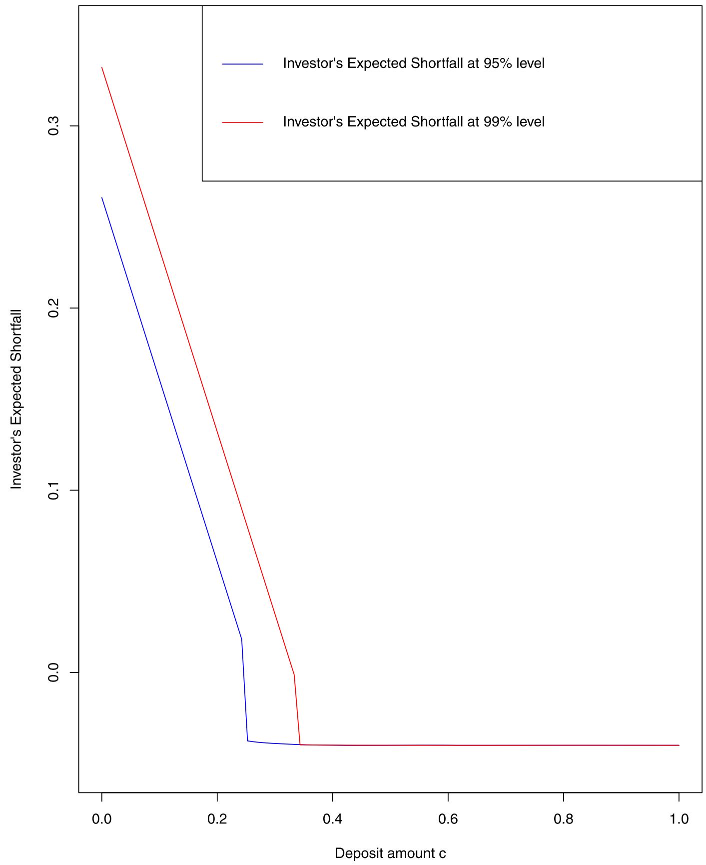

Fig. 10. Expected shortfall of the investor's losses as a function of the manager's deposit c. $X_0 = 1, T = 1, \alpha = 20\%, p = 0.01, q = 0.05, \mu_1 = 0.1, \mu_2 = -0.05, \sigma_1 = 0.1, \sigma_2 = 0.2, N_{sim} = 1{,}000{,}000, H = 0.04$.

folio, and the investor, in terms of the decision to withdraw from the fund, could both be studied, either in isolation (as a stochastic control problem and an optimal stopping problem respectively), or together (in a stochastic game of control and stopping). Finally, the limitations of the assumptions underlying risk-neutral valuation (particularly the ability to observe the value of, and dynamically trade in, the underlying assets of the hedge fund) could be investigated, perhaps through models that more realistically represent the bargaining process between principal (investor) and agent (manager) that we have discussed this paper.

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